Continuity and differentiability

Given the following function:

$$f(x,y) = \begin{cases} \frac{yx^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

Analyze the differentiability at the origin, justifying in detail.

Solution

To analyze the differentiability at $(0,0)$, we follow these steps:

- Check if the function is continuous at $(0,0)$.
- Compute the partial derivatives at $(0,0)$.
- Verify if the function is differentiable at $(0,0)$ using the definition.

Continuity at $(0,0)$. We want to determine the double limit of $f(x,y)$ as $(x,y)\to(0,0)$ by expressing it as the product of an infinitesimal function and a bounded function, and then applying the theorem of infinitesimal times bounded function.

We can rewrite $f(x,y)$ for $(x,y)\neq(0,0)$ as:

$$f(x,y) = y\left(\frac{x^2}{x^2+y^2}\right)$$

Consider the function:

$$g(x,y) = \frac{x^2}{x^2+y^2}$$

We will show that $g(x,y)$ is bounded between 0 and 1 for all $(x,y)\neq(0,0)$.

- Since $x^2\geq 0$ and $y^2\geq 0$, we have $x^2+y^2>0$ for $(x,y)\neq(0,0)$.
- Therefore, $g(x,y)\geq 0$.
- Also, $x^2+y^2\geq x^2$ implies:

$$g(x,y) = \frac{x^2}{x^2+y^2} \leq \frac{x^2}{x^2} = 1$$

Thus:

$$0\leq g(x,y)\leq 1$$

As $(x,y)\to(0,0)$, the term y approaches zero. Therefore, y is an infinitesimal.

Since $g(x,y)$ is bounded and y tends to zero, the product $y\cdot g(x,y)$ tends to zero:

$$\lim_{(x,y)\to(0,0)} y\cdot g(x,y) = 0$$

Therefore:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} y\cdot\left(\frac{x^2}{x^2+y^2}\right) = 0$$

By expressing $f(x,y)$ as the product of an infinitesimal y and a bounded function $g(x,y)$, we have shown that:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

However, since $f(0,0)=1$, we have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \neq f(0,0) = 1$$

Conclusion: The function $f(x,y)$ is not continuous at $(0,0)$.

A function f is differentiable at $(0,0)$ if there exists a tangent plane at that point, meaning it can be approximated by a linear function in a neighborhood of $(0,0)$.

Since f is not continuous at $(0,0)$, it is not differentiable at that point.

Conclusion: The function $f(x,y)$ is not differentiable at $(0,0)$ because it is not continuous at that point.